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The refraction of head seas by a long ship

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It is known that head seas cannot travel without deformation along a horizontal cylinder of full constant cross-section. Calculations are given which indicate that the waves are refracted away from the axis of the cylinder. Similar refraction effects are found for waves generated by a pulsating source on the cylinder, and also for the Kelvin wave pattern generated by a long ship of nearly constant cross-section moving with constant speed in the axial direction.

1. Introduction

In an earlier paper (Ursell 1968*a*, hereafter referred to as II) it was shown that head seas cannot travel along a long cylindrical ship without deformation. The first attempt to find this deformation was made in II, § 5, where the deformation along a thin wedge-like ship of great but finite length was considered. It was found that the amplitude of the diffracted wave near the wedge ultimately increases like $(Kx)^{\frac{1}{2}}$, where x is the distance from the bow along the ship; this high amplitude is confined to a horizontal layer near the ship which increases in width like $(Kx)^{\frac{3}{4}}$. The total wave amplitude is the amplitude of the sum of the incident head sea and the diffracted wave, and depends on their relative phase. Arguments can be given which tend to show that the relative phase in this case depends on the wave motion near the bow, but not on the shape of the cross-section of the wedge.

A long ship of full section was considered by Faltinsen (1973), who used slender-body theory and matched asymptotic expansions. He found that the amplitude of the diffracted wave near the ship is ultimately equal and opposite to that of the incident head sea, in a layer increasing like $(Kx)^{\frac{1}{2}}$. Thus the total wave amplitude near a ship of full section tends ultimately to zero; we may say that the incident wave is *refracted away from the ship*. (The effect of forward speed was also considered by Faltinsen, but for the sake of simplicity it will not be considered in problems 1 and 2 of the present paper.)

The arguments given in these papers were not conclusive, either for the thin ship or for the ship of full section, but they were plausible. The differences between the results in the two cases remain to be reconciled. In the present paper an attempt will be made to provide further evidence. The ship will be replaced by an infinitely long horizontal cylinder of constant cross-section on which the normal velocity is suitably prescribed. (The same idea will also be used to treat the refraction of the steady Kelvin pattern away from a long ship moving with constant forward speed.) For the sake of simplicity the cross-section will be taken to be a half-immersed circle, which can be treated comparatively simply, but the arguments of the present work can be generalized to an arbitrary constant cross-section.

Three problems to which the linearized theory is applicable will be treated.

Problem 1. Zero forward speed is assumed. On the semi-infinite stern section of the cylinder a wavelike pulsating normal velocity is prescribed while on the semi-infinite bow section the normal velocity vanishes. The resulting wave motion near the stern section is to be found; it is reasonable to hope that this motion will resemble the diffracted wave due to a semi-infinite ship. When the incident-wave potential is $e^{-Kz}\,e^{iKx-i\sigma t}$ the principal wave component of the total (incident plus diffracted) wave motion near the stern section will be found to be

$$\phi_{1,K}(x,y,z)\,e^{-i\sigma t} = -\frac{1}{\pi}\,e^{-\frac{1}{4}i\pi}\left(\frac{1}{2\pi Kx}\right)^{\frac{1}{2}}\Phi_*(K,y,z)\,e^{iKx-i\sigma t},$$

where Φ_* is the potential defined in equations (2.11)–(2.14) below.

Problem 2. Zero forward speed is again assumed. A pulsating normal velocity of constant frequency is prescribed over a finite part of the cylinder, and the wave motion near the stern section is to be found. The principal wave component will be found to be of the form

$$\phi_{2,K}(x,y,z)\,e^{-i\sigma t} = (A_2/x^{\frac{3}{2}})\,\Phi_*(K,y,z)\,e^{iKx-i\sigma t},$$

where the constant A_2 depends on the details of the prescribed velocity distribution.

Problem 3. A constant non-zero forward speed U is assumed, and the motion is steady. The normal velocity is prescribed over a finite part of the cylinder, and the refraction of the transverse waves of the Kelvin wave pattern along the stern section is to be found. The principal wave component will be found to be of the form

$$\phi_{3,K_0}(x,y,z) = (A_3/x^{\frac{3}{2}})\,\Phi_*(K_0,y,z)\cos(K_0x+\epsilon_3),$$

where $K_0 = g/U^2$ and where the constants A_3 and ϵ_3 depend on the details of the prescribed velocity distribution.

It will emerge that the same mathematical technique is applicable to all three problems. Even with our restrictive assumptions a complete solution is not feasible, but much information can be obtained about the wave motion when $x \to +\infty$, in the stern direction.

2. Mathematical preliminaries

Let the x axis be horizontal and along the axis of the cylinder, the y axis horizontal and normal to the x axis, and the z axis vertical (z increasing with depth). Also let cylindrical polar co-ordinates be defined by the equations $y = r\sin\theta$ and $z = r\cos\theta$; then on the immersed part C of the cylinder we have $r = a$, $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$ and $-\infty < x < \infty$. It will be seen that the Fourier transform

$$\Phi(k,y,z) = \int_{-\infty}^{\infty}\phi(x,y,z)\,e^{-ikx}\,dx \tag{2.1}$$

in each problem satisfies a differential equation of the form

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k^2\right)\Phi(k,y,z)=0 \tag{2.2}$$

in the fluid, and boundary conditions

$$\partial\Phi/\partial r = V(k,\theta) \quad \text{on} \quad C \tag{2.3}$$

and $$\partial\Phi/\partial z + F(k)\,\Phi = 0 \quad \text{on the free surface} \quad z=0, \quad r>a, \tag{2.4}$$

where $V(k,\theta)$ is a prescribed even function of θ, and $F(k)$ is a known positive function of k. In the present section only, let us write

$$F(k) = \begin{cases} |k|\cosh\gamma(k), & 0<\gamma<\infty, \quad \text{when} \quad F(k)>|k|, \\ |k|\cos\gamma_1(k), & 0<\gamma_1<\tfrac{1}{2}\pi, \quad \text{when} \quad F(k)<|k|. \end{cases} \tag{2.5}$$

Let us consider first a value of k for which $F(k)>|k|$. Then it is known (Ursell 1968*b*, hereafter referred to as III) that $\Phi(k,y,z)$ can be expanded in a series of the form

$$\Phi(k,y,z) = p_0(k)\,\Psi_0(|k|,y,z,\gamma) + \sum_{m=1}^{\infty} p_{2m}(k)\frac{\Psi_{2m}(|k|,y,z,\gamma)}{K'_{2m}(|k|\,a)}, \tag{2.6}$$

where

$$\Psi_0(|k|,y,z,\gamma) = 2\int_0^\infty \frac{\cosh\mu}{\cosh\mu-\cosh\gamma}\exp(-|k|\,z\cosh\mu)\cos(|k|y\sinh\mu)\,d\mu \tag{2.7}$$

(with an appropriate path of integration avoiding the pole $\mu=\gamma$) is a source function and the functions

$$\Psi_{2m}(|k|,y,z,\gamma) = K_{2m}(|k|\,r)\cos 2m\theta + 2\cosh\gamma(k)\,K_{2m-1}(|k|\,r)\cos(2m-1)\theta + K_{2m-2}(|k|\,r)\cos(2m-2)\theta, \quad m=1,2,3,\ldots, \tag{2.8}$$

are wave-free potentials. It is important to determine the correct path of integration in (2.7), i.e. the appropriate radiation condition for large $|y|$. This is found as follows. Let a small positive Rayleigh viscosity ϵ be introduced, then the free-surface boundary condition becomes

$$\partial\Phi_\epsilon/\partial z + F_\epsilon(k)\,\Phi_\epsilon = 0 \quad \text{on} \quad z=0, \quad r>a,$$

where $F_\epsilon(k)$ is complex valued and $F_\epsilon(k)\to F(k)$ when $\epsilon\to 0+$ (i.e. when $\epsilon\to 0$ through positive values). We write $F_\epsilon(k)=|k|\cosh\gamma_\epsilon(k)$; the path for $\Psi_0(|k|,y,z,\gamma_\epsilon)$ is the positive real μ axis as long as $\epsilon>0$. Now let $\epsilon\to 0+$, then the pole $\gamma_\epsilon(k)$ of the integrand tends to $\gamma(k)$ on the real μ axis. The appropriate path of integration in (2.7) is now chosen by the following self-evident rule: if it is found that $\gamma_\epsilon(k)\to\gamma(k)$ from below (above) then the path of integration in (2.7) is taken to pass above (below) $\mu=\gamma$. The corresponding integral will be denoted by Ψ_0^+ (Ψ_0^-).

Let us consider next a value of k for which $F(k)<|k|$. The denominator in (2.7) is then $\cosh\mu-\cos\gamma_1$ and is regular on the whole of the positive real μ axis, which is therefore taken as the path of integration. The corresponding integral will be denoted by Ψ_0. The wave-free potentials are defined as in (2.8), with $\cos\gamma_1(k)$ in place of $\cosh\gamma(k)$. In any given problem the functions $\gamma(k)$ and $\gamma_1(k)$ are known functions of k, but it will often be convenient to retain notations like $\Psi(|k|,y,z,\gamma)$ in order to exhibit the dependence on γ.

It will be seen later that the wavenumbers of greatest interest for the flow near the stern satisfy the equation $F(k) = |k|$, i.e. $\gamma = 0$. Then it is known (cf. III, §5) that the potential $\Phi(k,y,z)$ can be expanded in the form

$$\Phi(k,y,z) = p_{00}(k)\,\Psi_{00}(|k|,y,z,0)+p_{01}(k)\,e^{-|k|z}+\sum_{m=1}^{\infty} p_{2m}(k)\,\frac{\Psi_{2m}(|k|,y,z,0)}{K'_{2m}(|k|a)}, \tag{2.9}$$

where

$$\Psi_{00}(|k|,y,z,0) = \frac{1}{2}\left(\int_u+\int_l\right)\frac{\cosh\mu}{\cosh\mu-1}\exp\left(-|k|\,z\cosh\mu\right)\cos\left(|k|\,y\sinh\mu\right)d\mu, \tag{2.10}$$

where the integration limits are $\pm\infty$ with the path of integration of the first (second) integral passing below (above) the double pole $\mu = 0$. It will be observed that the expansion (2.9) contains one more coefficient than the expansion (2.6). It was shown in II that there is a unique potential $\Phi_*(|k|,y,z)$ satisfying

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-|k|^2\right)\Phi_*(|k|,y,z) = 0 \quad \text{in the fluid,} \tag{2.11}$$

$$\partial\Phi_*/\partial r = 0 \quad \text{on} \quad r = a, \quad -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi, \tag{2.12}$$

$$\partial\Phi_*/\partial z+|k|\,\Phi_* = 0 \quad \text{on} \quad z = 0, \quad r > a, \tag{2.13}$$

with an expansion of the form (2.9) normalized so that $p_{00} = 1$. This potential is unbounded for large $|y|$,

$$\Phi_*(|k|,y,z) \sim -2\pi\,|ky|\,e^{-|k|z} \quad \text{as} \quad |ky|\to\infty, \tag{2.14}$$

and therefore has no immediate physical interpretation. Nevertheless, it will be seen (as has already been mentioned in the introduction) that the potential $\Phi_*(|k|,y,z)$ plays an important part in describing the waves near the stern section $x\to+\infty$ when $|ky|$ is not large.

We shall need the following expansions (Ursell 1962, hereafter referred to as I):

$$\Psi_0^{\pm}(|k|,y,z,\gamma) = \pm\,2\pi i\coth\gamma\,\mathscr{T}(|k|,y,z,\gamma)+2\mathscr{R}(|k|,y,z,\gamma), \tag{2.15}$$

where $\mathscr{T}(|k|,y,z,\gamma) = \exp\left(-|k|\,z\cosh\gamma\right)\cos\left(|k|\,y\sinh\gamma\right)$,

$$\mathscr{R}(|k|,y,z,\gamma) = -\gamma\coth\gamma\exp(-|k|z\cosh\gamma)\cos\left(|k|y\sinh\gamma\right)$$
$$\times K_0(|k|\,r)+2\sum_{m=1}^{\infty}(-1)^{m-1}\left[\frac{\partial}{\partial\nu}\left(I_\nu(|k|\,r)\cos\nu\theta\right)\right]_{\nu=m}\sinh m\gamma\coth\gamma$$

are regular functions of γ near $\gamma = 0$; see I, equation (2.13). (It is important to note that $\Psi_0^{\pm}\to\infty$ as $\gamma\to 0$.) When $\gamma = i\gamma_1$ the corresponding expansion is

$$\Psi_0(|k|,y,z,i\gamma_1) = 2\pi\cot\gamma_1\,\mathscr{T}(|k|,y,z,i\gamma_1)+2\mathscr{R}(|k|,y,z,i\gamma_1).$$

It can also be shown (II, equation A.1.2) that

$$\Psi_{00}(|k|,y,z,0) = 2\mathscr{R}(|k|,y,z,0). \tag{2.16}$$

We shall also write

$$\langle a\,\partial\Psi_0^{\pm}/\partial r\rangle_{r=a} = \pm\,2\pi i\coth\gamma\,T(|k|,\theta,\gamma)+2R(|k|,\theta,\gamma) \tag{2.17}$$

and

$$\langle a\,\partial\Psi_0/\partial r\rangle_{r=a} = 2\pi\cot\gamma_1\,T(|k|,\theta,i\gamma_1)+2R(|k|,\theta,i\gamma_1), \tag{2.18}$$

where the functions T and R are obtained from $\mathscr{T}$ and $\mathscr{R}$ by differentiation and are evidently regular near $\gamma = 0$. Angular brackets will be used to indicate that r is to be put equal to a after differentiation.

By use of expansions such as (2.6) the function $\Phi(k, y, z)$ can in principle be found for all real k. If it could be found explicitly we could then infer the velocity potential $\phi(x, y, z)$ everywhere by means of the inverse Fourier transform

$$\phi(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(k,y,z)\, e^{ikx}\, dk. \tag{2.19}$$

Unfortunately this will not be found possible. We shall however be mainly concerned with the form of the waves near the stern section, i.e. the region where y and z are bounded and $x \to +\infty$. Asymptotic techniques are then applicable to (2.19). The principal contributions come from values of k at which $\Phi(k, y, z)$ or one of its derivatives ceases to be a regular function of k (cf. Lighthill 1958, chap. 4). On physical grounds we expect the dominant wavenumber to be $K = \sigma^2/g$ in problems 1 and 2 and $K_0 = g/U^2$ in problem 3, and it will be seen that at these wavenumbers the function $\Phi(k, y, z)$ is indeed not regular, because the condition $F(k) = |k|$ is satisfied. It will be assumed that $\Phi(k, y, z)$ is regular at all other wavenumbers except $k = 0$. (The contribution from $k = 0$ is evidently not wavelike and will not be considered further.)

3. Problem 1. The action of a fixed long ship on head seas

Let us begin by considering a fixed ship of great but finite length, and let us suppose that its parallel middle-body is a horizontal cylinder of semicircular cross-section. Let a regular wave train

$$\phi_{\text{inc}}\, e^{-i\sigma t} = e^{-Kz}\, e^{iKx - i\sigma t} \tag{3.1}$$

be incident on the ship from the negative x direction. The potential $\phi_{\text{diff}}\, e^{-i\sigma t}$ of the diffracted wave then satisfies

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi_{\text{diff}}(x,y,z) = 0 \quad \text{in the fluid,} \tag{3.2}$$

with the boundary conditions

$$\partial\phi_{\text{diff}}/\partial n = -\partial(e^{-Kz}\, e^{iKx})/\partial n \quad \text{on the ship} \tag{3.3}$$

and $$(K + \partial/\partial z)\,\phi_{\text{diff}} = 0 \quad \text{on the mean free surface} \quad z = 0, \tag{3.4}$$

where $K = \sigma^2/g$. There is also a radiation condition, which states that ϕ_{diff} represents outward-travelling waves at infinity. For a derivation of these equations see Lamb (1932, § 227). The factor $e^{-i\sigma t}$ will henceforth be omitted in this and the following section.

The boundary-value problem which has just been formulated cannot be readily solved, and we therefore now replace it by a simpler problem relating to a simpler body. The long but finite ship is replaced by a horizontal cylinder of uniform semi-circular cross-section extending from $x = -\infty$ to $x = +\infty$. We consider a potential $\phi_1(x,y,z)$ satisfying the equation of continuity (3.2) and the free-surface condition (3.4). The boundary condition on the cylinder is taken to be

$$\partial\phi_1/\partial r = -e^{iKx}h(x)\,\partial e^{-Kz}/\partial r \quad \text{on} \quad r = a, \tag{3.5}$$

where
$$h(x) = \begin{cases} 0 & \text{on the forward part} \quad -\infty < x \leqslant -l, \\ 1 & \text{on the near part} \quad l \leqslant x < \infty, \end{cases}$$

and where $h(x)$ is chosen to be an infinitely differentiable increasing function on the middle part $-l \leqslant x \leqslant l$. It will be seen later that our results are independent of the precise form of $h(x)$. For large positive x we may then reasonably expect ϕ_1 to behave like the diffracted wave from a semi-infinite (or a long but finite) ship.

To solve this boundary-value problem, a Fourier transform with respect to x is used. Let us write

$$\Phi_1(k,y,z) = \int_{-\infty}^{\infty} \phi_1(x,y,z)\,e^{-ikx}\,dx. \tag{3.6}$$

Then Φ_1 is defined in the fluid region ($r > a$, $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$) of the y,z plane. We evidently have

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-k^2\right)\Phi_1(k,y,z) = 0 \quad \text{in the fluid,} \tag{3.7}$$

with the boundary conditions

$$\frac{\partial\Phi_1}{\partial r} = -\left(\frac{\partial}{\partial r}e^{-Kz}\right)\int_{-\infty}^{\infty} h(x)\,e^{-ix(k-K)}\,dk = -\left(\frac{\partial}{\partial r}e^{-Kz}\right)H(k-K), \quad \text{say,} \quad \text{on} \quad r = a \tag{3.8}$$

and
$$\partial\Phi_1/\partial z + K\Phi_1 = 0 \quad \text{on the free surface} \quad z = 0, \quad r > a. \tag{3.9}$$

Let us write
$$K = \begin{cases} |k|\cosh\alpha(k) & \text{when} \quad |k| < K, \quad (3.10) \\ |k|\cos\alpha_1(k) & \text{when} \quad |k| > K, \quad (3.11) \end{cases}$$

cf. (2.5) above. As was explained in § 2, we must now find the correct path of integration for the source function Ψ_0. When the small Rayleigh viscosity ϵ is introduced, we find by a simple calculation (which is omitted) that the free-surface boundary condition for $\Phi_{1\epsilon}(k,y,z)$ becomes

$$\partial\Phi_{1\epsilon}/\partial z = -(K+\epsilon i\sigma/g)\,\Phi_{1\epsilon}. \tag{3.12}$$

The parameter $\alpha_\epsilon(k)$ is determined from the equation

$$|k|\cosh\alpha_\epsilon(k) = K+\epsilon i\sigma/g.$$

It is evident that $\alpha_\epsilon(k) \to \alpha(k)$ from *above* as $\epsilon \to 0+$, thus the appropriate source function is Ψ_0^- for all wavenumbers k satisfying $|k| < K$. We now follow the procedure of § 2, in order to determine the analytic behaviour of $\Phi(k,y,z)$ near $k = K$.

As we have just seen, in the range $0 < k < K$ we have

$$\Phi_1(k,y,z) = p_0(k)\,\Psi_0^-(k,y,z) + \sum_{m=1}^{\infty}\frac{p_{2m}(k)}{K'_{2m}(ka)}\Psi_{2m}(k,y,z), \tag{3.13}$$

where from the boundary condition (3.5) on the circle we must have

$$-H(k-K)\left\langle a\frac{\partial}{\partial r}e^{-Kz}\right\rangle = p_0(k)\left\langle a\frac{\partial \Psi_0}{\partial r}\right\rangle + \sum_{m=1}^{\infty}\frac{p_{2m}(k)}{K'_{2m}(ka)}\left\langle a\frac{\partial \Psi_{2m}}{\partial r}\right\rangle, \quad (3.14)$$

i.e.

$$2\pi i\coth\alpha\, T(k,\theta) + 2R(k,\theta) = -\frac{H(k-K)}{p_0(k)}\left\langle a\frac{\partial}{\partial r}e^{-Kz}\right\rangle - \Sigma\frac{p_{2m}(k)}{p_0(k)}\frac{1}{K'_{2m}(ka)}\left\langle a\frac{\partial \Psi_{2m}}{\partial r}\right\rangle \quad \text{when} \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi. \quad (3.15)$$

This may be regarded as an expansion of the left-hand side of (3.15) in terms of the set of functions in angular brackets on the right-hand side of (3.15). The latter functions are evidently regular functions of θ and also of k near K, but we note that $\coth\alpha = K/(K^2-k^2)^{\frac{1}{2}}$ has a branch point at $k = K$, while $H(k-K)$ has a simple pole [see equation (3.20) below]. We also note that the functions

$$\langle a\,\partial e^{-Kz}/\partial r\rangle = -Ka\cos\theta\exp(-Ka\cos\theta)$$

and

$$\begin{aligned} T(k,\theta) &= a\partial[\exp(-kr\cos\theta\cosh\alpha)\cos(kr\sin\theta\sinh\alpha)]/\partial r \\ &= -Ka\cos\theta\exp(-Ka\cos\theta)\cos\{(K^2-k^2)^{\frac{1}{2}}a\sin\theta\} \\ &\quad -(K^2-k^2)^{\frac{1}{2}}a\exp(-Ka\cos\theta)\sin\theta\sin\{(K^2-k^2)^{\frac{1}{2}}a\sin\theta\} \end{aligned}$$

are nearly equal when k is near K. It is in fact not difficult to see that, near $k = K$,

$$T(k,\theta) = \langle a\,\partial e^{-Kz}/\partial r\rangle + O(K-k). \quad (3.16)$$

Thus, near $k = K$, (3.15) can be rewritten in the form

$$2\pi i\coth\alpha\{\langle a\,\partial e^{-Kz}/\partial r\rangle + O(K-k)\} + 2R(k,\theta) = -\frac{H(k-K)}{p_0}\left\langle a\frac{\partial}{\partial r}e^{-Kz}\right\rangle - \Sigma\frac{p_{2m}}{p_0}\frac{1}{K'_{2m}(ka)}\left\langle a\frac{\partial \Psi_{2m}}{\partial r}\right\rangle. \quad (3.17)$$

We can now (on the assumption that Ka is neither small nor large) make inferences about the analytic form of the coefficients. We have

$$\left.\begin{aligned} -H(k-K)/p_0 &= 2\pi i\coth\alpha + 2R_0(k) + O(K-k)^{\frac{1}{2}} \\ \text{and} \qquad -p_{2m}/p_0 &= 2R_{2m}(k) + O(K-k)^{\frac{1}{2}}, \end{aligned}\right\} \quad (3.18)$$

where $R_0(k)$ and $R_{2m}(k)$ are the coefficients in the expansion

$$R(k,\theta) = R_0(k)\left\langle a\frac{\partial}{\partial r}e^{-Kz}\right\rangle + \Sigma R_{2m}(k)\frac{1}{K'_{2m}(ka)}\left\langle a\frac{\partial \Psi_{2m}}{\partial r}\right\rangle, \quad (3.19)$$

valid in the range $0 \leqslant \theta \leqslant \frac{1}{2}m$. We must still find the behaviour of $H(k-K)$ near $k = K$. We have

$$\begin{aligned} H(k-K) &= \int_{-l}^{\infty} h(x)\,e^{-ix(k-K)}\,dx \\ &= \left[h(x)\frac{e^{i(K-k)x}-1}{i(K-k)}\right]_{x=-l}^{x=\infty} - \int_{-l}^{l} h'(x)\frac{e^{i(K-k)x}-1}{i(K-k)}\,dx, \\ &\qquad \text{by integration by parts,} \\ &= \frac{i}{K-k} - \int_{-l}^{l} xh'(x)\,dx + O(K-k), \end{aligned} \quad (3.20)$$

since $h'(x) = 0$ when $|x| > l$, where $H(k-K)$ has been interpreted by means of the Rayleigh viscosity as $\lim_{\epsilon\to 0+} H(k-K-\epsilon i\sigma/g)$. We can now find the form of the potential

$$\begin{aligned}\Phi_1(k,y,z) &= p_0(k)\{2\pi i \coth\alpha\,\mathscr{T}(k,y,z)+2\mathscr{R}(k,y,z)\}+\Sigma\frac{p_{2m}(k)}{K'_{2m}(ka)}\Psi_{2m}(k,y,z)\\ &= -\frac{2\pi i\coth\alpha\,\mathscr{T}+2\mathscr{R}}{2\pi i\coth\alpha+2R_0+O(K-k)^{\frac{1}{2}}}H(k-K)\\ &\qquad+\Sigma\frac{2R_{2m}+O(K-k)^{\frac{1}{2}}}{2\pi i\coth\alpha+2R_0+O(K-k)^{\frac{1}{2}}}\frac{H(k-K)}{K'_{2m}(ka)}\Psi_{2m}(k,y,z).\end{aligned} \tag{3.21}$$

We observe that

$$\Psi_{2m}(k,y,z) = \Psi_{2m}(K,y,z)+O(K-k);$$

corresponding relations hold for the regular functions $\mathscr{T}$, $\mathscr{R}$, K'_{2m} and R_{2m}. Thus, on expanding the quotients in (3.21), we find that, near $k = K$, the potential has the form

$$\begin{aligned}\Phi_1(k,y,z) = -\frac{i}{K-k}\mathscr{T}(K,y,z)-\frac{\tanh\alpha}{\pi(K-k)}\{\mathscr{R}(K,y,z)-R_0(K)\mathscr{T}(K,y,z)\}\\ +\frac{\tanh\alpha}{\pi(K-k)}\Sigma\frac{R_{2m}(K)\Psi_{2m}(K,y,z)}{K'_{2m}(Ka)}+O(1).\end{aligned}$$

On substituting $\mathscr{T}(K,y,z) = e^{-Kz}$ and

$$\tanh\alpha = (K^2-k^2)^{\frac{1}{2}}/K = (2/K)^{\frac{1}{2}}(K-k)^{\frac{1}{2}}+O(K-k)^{\frac{3}{2}},$$

this becomes

$$\Phi_1(k,y,z) = -\frac{i\,e^{-Kz}}{K-k}-\frac{1}{\pi}\left(\frac{1}{2K(K-k)}\right)^{\frac{1}{2}}\Phi_*(K,y,z)+O(1) \quad\text{when}\quad k<K,$$

where the potential

$$\Phi_*(K,y,z) = 2\mathscr{R}(K,y,z)-2R_0(K)\mathscr{T}(K,y,z)-2\Sigma\frac{R_{2m}(K)}{K'_{2m}(Ka)}\Psi_{2m}(K,y,z) \tag{3.22}$$

is identical with the potential defined by (2.11)–(2.14). To show this, it is sufficient to note that every term on the right-hand side of (3.22) satisfies the wave equation (2.11) in the fluid, and the free-surface condition (2.13). Also the boundary condition $\langle a\,\partial\Phi_*/\partial r\rangle = 0$ is satisfied on account of (3.19); furthermore, as $Ky\to\infty$ in (3.22),

$$2\mathscr{R}(K,y,z) = \Psi_{00}(K,y,z,0) \sim -2\pi K|y|\,e^{-Kz},$$

while all the other terms in (3.22) are of smaller magnitude. These are the conditions which uniquely defined $\Phi_*(K,y,z)$ in § 2 above.

Equation (3.22) describes the behaviour of $\Phi_1(k,y,z)$ near $k = K$ when $k < K$. Similarly, we can find the behaviour when $k > K$. It is only necessary to replace $\Psi_0^-(k,y,z,\alpha)$ by $\Psi_0(k,y,z,i\alpha_1)$, where $\tan\alpha_1 = (k^2-K^2)^{\frac{1}{2}}/K$; in other words, to replace $(K-k)^{\frac{1}{2}}$ by $-i(k-K)^{\frac{1}{2}}$. Thus

$$\Phi_1(k,y,z) = -\frac{i\,e^{-Kz}}{K-k}+\frac{i}{\pi}\left(\frac{1}{2K(k-K)}\right)^{\frac{1}{2}}\Phi_*(K,y,z)+O(1) \quad\text{when}\quad k>K. \tag{3.23}$$

We can now use these results in the inversion formula

$$\phi_1(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_1(k,y,z)\,e^{ikx}\,dk. \tag{3.24}$$

As was explained in § 2 above, the singular behaviour of Φ_1 at $k = K$ gives rise to a term, $\phi_{1K}(x, y, z)$ say, in the asymptotic expansion of ϕ_1 for large positive x. We find that

$$\phi_{1K}(x,y,z) = -e^{-Kz}e^{iKx} - \frac{1}{\pi}e^{-\frac{1}{4}i\pi}\left(\frac{1}{2\pi Kx}\right)^{\frac{1}{2}} e^{iKx}\,\Phi_*(K,y,z) + e^{iKx}O(x^{-\frac{3}{2}}), \quad (3.25)$$

where the first term is inferred from (3.20) and the second term is obtained from the integrals

$$\int_0^\infty \frac{1}{v^{\frac{1}{2}}}e^{\pm iv}\,dv = \pi^{\frac{1}{2}}e^{\pm\frac{1}{4}i\pi}.$$

The first term in (3.25) represents that regular wave train which has precisely the normal velocity (3.3) on the cylinder $r = a$ when $x \geqslant l$. The second term represents a wave decaying in the x direction. (The contribution from $k = -K$ can be shown to be of smaller magnitude.)

The result (3.25) is independent of the function $h(x)$ appearing in (3.5). It is therefore reasonable to suppose that a *semi-infinite* body consisting of a circular cylinder from $x = l$ to $x = \infty$ together with a smooth bow section will have the same asymptotic field if the prescribed normal velocity is the same. Let a wave train $\phi_{\text{inc}} = e^{-Kz}e^{iKx}$ be incident on such a body. The total wave field near the body for large x would then be expected to be

$$\phi_{\text{inc}} + \phi_{1K} \sim -\frac{1}{\pi}e^{-\frac{1}{4}i\pi}\left(\frac{1}{2\pi Kx}\right)^{\frac{1}{2}} e^{iKx}\Phi_*(K,y,z). \quad (3.26)$$

The wave amplitude thus tends to zero for fixed y and z as $x \to +\infty$. In other words, the incident wave train is *refracted away* from the semi-infinite cylinder, leaving a comparatively wave-free zone near the body.

What is the width of this zone? We would expect it to increase in width as x increases. Is the ultimate width finite or infinite? If it were finite, then for (y, z) outside this zone we would have $\phi_{\text{inc}} + \phi_{1K} \sim e^{-Kz}e^{iKx}$, but (3.26) holds for any fixed (y, z). Thus we conclude that the width of the comparatively wave-free zone tends to ∞ as $x \to \infty$. Our method gives no information about the wave field in any other direction. For this the integral (3.24) would need to be evaluated for both x and y large, and this would depend on values of k not nearly equal to K. However, when $Kx \gg K|y| \gg 1$, the wave-free terms in (3.13) can be neglected, and the field can then be expressed in terms of Fresnel integrals. The calculation (which is omitted) shows that the width of the wave-free zone is of order $K^{-1}(Kx)^{\frac{1}{2}}$.

4. Problem 2. A distributed pulsating source on an infinite cylinder

We shall next study a potential $\phi_2(x, y, z)\,e^{-i\sigma t}$ satisfying the same equation of continuity (3.2) and the same free-surface condition (3.4) as ϕ_1, but satisfying on $r = a$ the boundary condition

$$\frac{\partial\phi_2}{\partial r} = \begin{cases} v(x,\theta) & \text{when} \quad |x| \leqslant l, \\ 0 & \text{when} \quad |x| > l, \end{cases} \quad (4.1)$$

where it is assumed for simplicity that $v(x, \theta)$ is an even function of θ. (This restriction can easily be removed.) The radiation condition at infinity, obtained by means of the Rayleigh viscosity, is evidently the same as in § 3. Let Φ_2 and V be defined by the equations

$$\Phi_2(k,y,z) = \int_{-\infty}^{\infty} \phi_2(x,y,z)\, e^{-ikx}\, dx, \tag{4.2}$$

$$V(k,\theta) = \int_{-\infty}^{\infty} v(x,\theta)\, e^{-ikx}\, dx. \tag{4.3}$$

Then, as in § 3 above, Φ_2 must have an expansion of the form

$$\Phi_2(k,y,z) = p_0^{(2)}\Psi_0^-(|k|,y,z,\alpha) + \sum_{m=1}^{\infty} \frac{p_{2m}^{(2)}(k)}{K'_{2m}(|k|a)} \Psi_{2m}(|k|,y,z,\alpha) \tag{4.4}$$

when $0 < k < K$, where

$$p_0^{(2)}\left\langle a\frac{\partial \Psi_0^-}{\partial r}\right\rangle + \Sigma \frac{p_{2m}^{(2)}}{K'_{2m}}\left\langle a\frac{\partial \Psi_{2m}}{\partial r}\right\rangle = aV(k,\theta) \quad \text{when} \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi. \tag{4.5}$$

It follows that

$$\left\langle a\frac{\partial \Psi_0^-}{\partial r}\right\rangle = \frac{aV(k,\theta)}{p_0^{(2)}} - \Sigma \frac{p_{2m}^{(2)}}{p_0^{(2)}}\frac{1}{K'_{2m}}\left\langle a\frac{\partial \Psi_{2m}}{\partial r}\right\rangle, \tag{4.6}$$

where the functions $V(k,\theta)$ and $\langle a\,\partial\Psi_{2m}/\partial r\rangle$ are regular functions of k near K. The analytic forms of $p_0^{(2)}$ and $p_{2m}^{(2)}$ can now be found in much the same way as in § 3 above. As before, the left-hand side of (4.6) is of the form

$$2\pi i \coth\alpha\{\langle a\,\partial\, e^{-Kz}/\partial r\rangle + O(K-k)\} + 2R(k,\theta), \tag{4.7}$$

and it follows that

$$\left.\begin{aligned} aV_0/p_0^{(2)} &= 2\pi i \coth\alpha\, T_0^{(2)} + 2R_0^{(2)} \\ \text{and} \qquad -p_{2m}^{(2)}/p_0^{(2)} &= 2\pi i \coth\alpha\, T_{2m}^{(2)} + 2R_{2m}^{(2)}, \end{aligned}\right\} \tag{4.8}$$

where V_0 is a real normalizing constant, and where $T_{2m}^{(2)}$ and $R_{2m}^{(2)}$ are the coefficients in the expansions

$$\left\langle a\frac{\partial}{\partial r}e^{-Kz}\right\rangle = \frac{V(k,\theta)}{V_0}T_0^{(2)} + \Sigma\, T_{2m}^{(2)}\frac{1}{K'_{2m}}\left\langle a\frac{\partial \Psi_{2m}}{\partial r}\right\rangle \tag{4.9}$$

and

$$R(k,\theta) = \frac{V(k,\theta)}{V_0}R_0^{(2)} + \Sigma\, R_{2m}^{(2)}\frac{1}{K'_{2m}}\left\langle a\frac{\partial \Psi_{2m}}{\partial r}\right\rangle. \tag{4.10}$$

Thus

$$\begin{aligned} \Phi_2(k,y,z) &= p_0^{(2)}(k)\{2\pi i \coth\alpha\,\mathscr{T} + 2\mathscr{R}\} + \Sigma\, p_{2m}^{(2)}\frac{1}{K'_{2m}(ka)}\Psi_{2m} \\ &= aV_0\frac{2\pi i\coth\alpha\,\mathscr{T} + 2\mathscr{R}}{2\pi i\coth\alpha\, T_0^{(2)} + 2R_0^{(2)}} - aV_0\,\Sigma\,\frac{2\pi i\coth\alpha\, T_{2m}^{(2)} + 2R_{2m}^{(2)}}{2\pi i\coth\alpha\, T_0^{(2)} + 2R_0^{(2)}}\frac{\Psi_{2m}}{K'_{2m}}, \end{aligned} \tag{4.11}$$

where $\mathscr{T}$, $\mathscr{R}$, $T_{2m}^{(2)}$ and $R_{2m}^{(2)}$ are regular functions of k near K, while

$$\tanh\alpha = (2/K)^{\frac{1}{2}}(K-k)^{\frac{1}{2}} + O(K-k)^{\frac{3}{2}}$$

has a branch point. On expanding the quotients in (4.11) and proceeding as in § 3 above, it is found that

$$\begin{aligned} \Phi_2(k,y,z) &= \text{regular function of } k \\ &\quad + \frac{aV_0\tanh\alpha}{\pi i T_0^{(2)}}\left[\mathscr{R} - \frac{R_0^{(2)}}{T_0^{(2)}}\mathscr{T} - \Sigma\left(R_{2m}^{(2)} - \frac{R_0^{(2)}}{T_0^{(2)}}T_{2m}^{(2)}\right)\frac{\Psi_{2m}}{K'_{2m}}\right]_{k=K} \\ &\quad + \text{smaller terms of order } k-K. \end{aligned} \tag{4.12}$$

It is now not difficult to show, as in § 3, that the expression in square brackets, Θ say, satisfies the wave equation (2.11) and the boundary condition (2.13) at the free surface, with $k = K$. Also it can be shown that $\partial\Theta/\partial r = 0$ on $r = a$. It follows that Θ is a multiple of Φ_*; in fact, $\Theta = \frac{1}{2}\Phi_*$, as can be seen by examining the behaviour for large $|y|$. Thus

$$\Phi_2(k,y,z) = \text{regular function of } k$$
$$+\frac{aV_0}{2\pi i T_0^{(2)}}\left(\frac{2}{K}\right)^{\frac{1}{2}}(K-k)^{\frac{1}{2}}\,\Phi_*(K,y,z)+\text{smaller terms} \quad \text{when } k<K. \tag{4.13}$$

When $k > K$, the second term of (4.13) must be replaced by

$$\frac{aV_0}{2\pi T_0^{(2)}}\left(\frac{2}{K}\right)^{\frac{1}{2}}(k-K)^{\frac{1}{2}}\,\Phi_*(K,y,z). \tag{4.14}$$

It may be noted that $V_0/T_0^{(2)}$ is the coefficient of $\langle a\,\partial e^{-Kz}/\partial r\rangle$ in the expansion

$$V(K,\theta) = \frac{V_0}{T_0^{(2)}}\left\langle a\frac{\partial}{\partial r}e^{-Kz}\right\rangle_{r=a} - \Sigma\frac{V_0T_{2m}^{(2)}}{T_0^{(2)}}\frac{1}{K'_{2m}(Ka)}\left\langle a\frac{\partial\Psi_{2m}}{\partial r}\right\rangle_{k=K}; \tag{4.15}$$

see (4.9) above. Thus the coefficient in (4.13) depends on

$$V(K,\theta) = \int_{-\infty}^{\infty} v(x,\theta)\,e^{-iKx}\,dx,$$

i.e. on the details of the prescribed velocity distribution.

It now follows that the leading term for large x in the expansion of

$$\phi_2(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_2(k,y,z)\,e^{ikx}\,dk \tag{4.16}$$

is

$$\phi_{2K}(x,y,z) = \frac{a}{4\pi^2}\frac{V_0}{T_0^{(2)}}\left(\frac{2}{K}\right)^{\frac{1}{2}}\Phi_*(K,y,z)$$
$$\times\left[-i\int_{-\infty}^{K}(K-k)^{\frac{1}{2}}\,e^{ikx}\,dk+\int_{K}^{\infty}(k-K)^{\frac{1}{2}}\,e^{ikx}\,dk\right]$$
$$= -\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}e^{-\frac{1}{4}i\pi}\,a\frac{V_0}{T_0^{(2)}}\frac{1}{K^{\frac{1}{2}}x^{\frac{3}{2}}}\,\Phi_*(K,y,z)\,e^{iKx}, \tag{4.17}$$

when y and z are kept fixed and $x \to +\infty$. Here the equations

$$\int_0^{\infty} v^{\frac{1}{2}}\,e^{\pm iv}\,dv = \tfrac{1}{2}\pi^{\frac{1}{2}}\,e^{\pm\frac{3}{4}i\pi}$$

have been used. (For the asymptotic treatment of Fourier integrals see Lighthill 1958, chap. 4.) Thus, apart from the obvious phase factor e^{iKx}, the distribution of pressure over each section x = constant is ultimately the same and is described by the function $\Phi_*(K,y,z)$ but the amplitude of the variation decreases like $x^{-\frac{3}{2}}$. (The contribution from $k = -K$ is of smaller magnitude.) The magnitude of the pressure is described by the proportionality factor $V_0/T_0^{(2)}$, which is the first coefficient in the expansion (4.15) above and depends on

$$V(K,\theta) = \int_{-\infty}^{\infty} v(x,\theta)\,e^{-iKx}\,dx$$

and thus on the prescribed velocity distribution. A lengthy calculation would be needed to determine $V_0/T_0^{(2)}$.

5. Problem 3. A long ship moving with constant forward speed U in still water

Let co-ordinate axes be fixed in the ship. In these co-ordinates the motion is steady, and a fixed Kelvin pattern is formed relative to the ship. As before, let the ship be replaced by an infinitely long cylinder of semicircular cross-section on which a distribution of normal velocity is prescribed. Then the velocity potential is of the form

$$\phi(x,y,z) = Ux+\phi_3(x,y,z), \tag{5.1}$$

where
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi_3(x,y,z) = 0$$

in the fluid, with the boundary conditions

$$\partial\phi_3/\partial r = v_3(x,\theta) \quad \text{on} \quad r=a \tag{5.2}$$

and
$$U^2\,\partial^2\phi_3/\partial x^2 = g\,\partial\phi_3/\partial z \quad \text{on the free surface} \quad z=0. \tag{5.3}$$

The latter condition is the well-known boundary condition of linearized steady thin-ship theory, but it is not difficult to see that the usual derivation (Havelock 1936) depends only on the decomposition (5.1), and that (5.3) is therefore applicable to the present problem. The appropriate radiation condition remains to be determined. Let a small positive Rayleigh viscosity ϵ be introduced. The free-surface boundary condition becomes

$$\left(U^2\frac{\partial^2}{\partial x^2}+U\epsilon\frac{\partial}{\partial x}\right)\phi_{3\epsilon} = g\frac{\partial\phi_{3\epsilon}}{\partial z} \quad \text{on} \quad z=0; \tag{5.4}$$

cf. Havelock (1936). Let the wavenumber K_0 be defined by

$$K_0 = g/U^2. \tag{5.5}$$

If we now define
$$\Phi_3(k,y,z) = \int_{-\infty}^{\infty}\phi_3(x,y,z)\,e^{-ikx}\,dx,$$

with a similar definition for $\Phi_{3\epsilon}$, then we have, on $z=0$,

$$\begin{aligned} g\frac{\partial\Phi_{3\epsilon}}{\partial z} &= \int_{-\infty}^{\infty} g\frac{\partial\phi_{3\epsilon}}{\partial z}e^{-ikx}\,dx \\ &= \int_{-\infty}^{\infty}\left(U^2\frac{\partial^2\phi_{3\epsilon}}{\partial x^2}+U\epsilon\frac{\partial\phi_{3\epsilon}}{\partial x}\right)e^{-ikx}\,dx \\ &= \int_{-\infty}^{\infty}U^2\phi_{3\epsilon}\frac{d^2}{dx^2}e^{-ikx}\,dx-\int_{-\infty}^{\infty}U\epsilon\phi_{3\epsilon}\frac{d}{dx}e^{-ikx}\,dx, \\ &\qquad\qquad \text{by integration by parts,} \\ &= -U^2k^2\Phi_{3\epsilon}+U\epsilon ik\Phi_{3\epsilon} = -gF_\epsilon(k)\,\Phi_{3\epsilon}, \quad \text{say,} \end{aligned} \tag{5.6}$$

where
$$F_\epsilon(k) = U^2k^2/g-U\epsilon ik/g.$$

We now proceed as in § 2 above. When $U^2k^2/g > |k|$, let a positive parameter β be defined by the equation

$$|k|\cosh\beta = U^2k^2/g = k^2/K_0,$$

and a parameter β_ϵ near β by the equation

$$|k| \cosh \beta_\epsilon = U^2k^2/g - \epsilon ik/g;$$

thus
$$\cosh \beta_\epsilon = \frac{|k|}{K_0} - \frac{U\epsilon i}{g}\frac{k}{|k|}. \tag{5.7}$$

When $\epsilon \to 0+$ it is easy to see that $\beta_\epsilon \to \beta$ from *below* when $k > K_0$, and the appropriate wave-source potential is therefore $\Psi_0^+(|k|, y, z, \beta)$, where β is the real positive root of $\cosh\beta = |k|/K_0$. We observe that $\tanh\beta = (k^2 - K_0^2)^{\frac{1}{2}}/|k|$ has a branch point when $k = K_0$. (Similarly, when $k < -K_0$, the appropriate wave-source potential is $\Psi_0^-(|k|, y, z, \beta)$.) Thus, when $k > K_0$, we write

$$\Phi_0(k, y, z) = p_0^{(3)}(k)\, \Psi_0^+(|k|, y, z, \beta) + \Sigma \frac{p_{2m}^{(3)}(k)}{K_m'(ka)} \Psi_{2m}(|k)|, y, z, \beta), \tag{5.8}$$

where

$$p_0^{(3)}\left\langle a\frac{\partial\Psi_0^+}{\partial r}\right\rangle + \Sigma \frac{p_{2m}^{(3)}}{K_{2m}'}\left\langle a\frac{\partial\Psi_{2m}}{\partial r}\right\rangle = aV(k,\theta) \quad \text{when} \quad r = a, \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi;$$

here
$$V_3(k,\theta) = -\int_{-\infty}^{\infty} v_3(x,\theta)\, e^{-ikx}\, dx.$$

The calculation is now identical with that of § 4 above, except that the factor $i \coth\alpha$ multiplying $\langle a\,\partial e^{-Kz}/\partial r\rangle$ in (4.7) must be replaced by $-i\coth\beta$. Thus, from (4.13), when $k > K_0$,

$$\Phi_3(k, y, z) = \text{regular function of } k$$
$$+ \frac{iaV_0}{2\pi T_0^{(3)}}\left(\frac{2}{K_0}\right)^{\frac{1}{2}} (k - K_0)^{\frac{1}{2}}\, \Phi_*(K_0, y, z) + \text{smaller terms.} \tag{5.9}$$

When $0 < k < K_0$, the calculation of § 4 must be modified by replacing $\cot\alpha_1$ by $\cot\beta_1$; it is then found that

$$\Phi_3(k, y, z) = \text{regular function of } k + \frac{aV_0}{2\pi T_0^{(3)}}\left(\frac{2}{K_0}\right)^{\frac{1}{2}} (K_0 - k)^{\frac{1}{2}}\, \Phi_*(K_0, y, z). \tag{5.10}$$

Similar calculations can be made when k is near $-K_0$, and then the leading terms in

$$\phi_3(x, y, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_3(k, y, z)\, e^{ikx}\, dk$$

can be found as in § 4 above. The contribution from k near K_0 is found to be

$$\phi_{3K_0}(x, y, z) = -\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} e^{\frac{1}{4}i\pi} \frac{aV_0}{T_0^{(3)}} \frac{1}{K_0^{\frac{1}{2}} x^{\frac{3}{2}}}\, \Phi_*(K_0, y, z)\, e^{iK_0x} \tag{5.11}$$

when $x \to +\infty$, where $V_0/T_0^{(3)}$ is the complex quantity in the expansion corresponding to (4.15), but with V_3 instead of V, $T_{2m}^{(3)}$ instead of $T_{2m}^{(2)}$, and K_0 instead of K. Similarly the contribution from k near $-K_0$ can be calculated, and is found to be the complex conjugate of (5.11); this was to be expected since $\phi_3(x, y, z)$ must be real. Thus, as $x \to +\infty$, while y and z remain fixed,

$$\phi_3(x, y, z) \sim -\frac{a}{2^{\frac{1}{2}}\pi^{\frac{1}{2}} K_0^{\frac{1}{2}}} \frac{\Phi_*(K_0, y, z)}{x^{\frac{3}{2}}} \operatorname{Re}\left(\frac{V_0}{T_0^{(3)}} \exp\left(iK_0x + \tfrac{1}{4}i\pi\right)\right).$$

When $x \to -\infty$, the potential is of much smaller magnitude.

6. Discussion and conclusions

In this paper we have been considering the effect of a long cylinder on waves travelling in the axial direction. In our first problem, which corresponds to incident head seas, the amplitude near the stern section decayed like $x^{-\frac{1}{2}}$, whereas head seas travelling along a plane vertical wall do not decay. In our second problem the waves were generated by a pulsating normal velocity and decayed like $x^{-\frac{3}{2}}$ in the axial direction; the corresponding decay along a plane vertical wall is like $x^{-\frac{1}{2}}$. In our third problem a Kelvin pattern was generated by a prescribed normal velocity component travelling with constant velocity and the transverse waves near the cylinder were found to decay like $x^{-\frac{3}{2}}$ whereas the Kelvin pattern near a plane wall (or in open water) decays like $x^{-\frac{1}{2}}$. In each of these three cases the decay is therefore more rapid than near a plane wall: the wave pattern is *refracted away from the cylinder*. The width of the comparatively wave-free zone tends to ∞ as x tends to ∞. These results were obtained for a cylinder of semicircular cross-section, but can be generalized to arbitrary (constant) cross-sections by formulating the problems in terms of integral equations, as in II. The result for our first problem is consistent with the results of Faltinsen (1973), which, as we now see, must represent the asymptotic behaviour of the waves when $x \to \infty$ in directions close to the axial direction. Faltinsen formulated his problem in terms of slender-body theory and matched asymptotic expansions.

We still have to discuss the thin-ship result obtained in II. It has just been noted that in problem 2 the amplitude decays like $x^{-\frac{3}{2}}$ along a cylinder and like $x^{-\frac{1}{2}}$ along a plane wall. It is reasonable to suppose that the rate of refraction depends on the wavelength as well as on the cross-section. When Ka is large, it may be conjectured that the refraction is small and only becomes effective at very large distances; when Ka is small or moderate, on the other hand, the effect of the curvature of the cross-section is felt even in the vicinity of the source. (Similar results would be expected in the other two problems.) It would be interesting to extend our calculation to large and to small values of Ka.

Our conclusions in each case depend on the analytical form of $\Phi(k, y, z)$ near a critical wavenumber. A typical equation is

$$\Phi_2(k, y, z) = \text{regular function of } k + \frac{aV_0}{2\pi T_0^{(2)}}\left(\frac{2}{K}\right)^{\frac{1}{2}}(k-K)^{\frac{1}{2}}\,\Phi_*(K, y, z) + \text{smaller terms}$$

when $k > K$. The simplicity of this expression is in contrast to the involved method of derivation, and a simpler derivation would be desirable.

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